UW Math Circle, Autumn 2016 Week 2

So, revel tie? (Again, mars ant...)

This week we talked about trees: we were interested in counting trees on $[n]$, counting how many spanning trees a given graph has, and how to find spanning trees.

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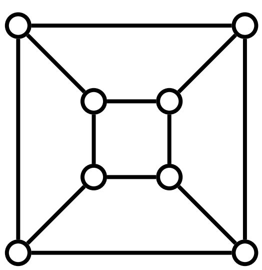
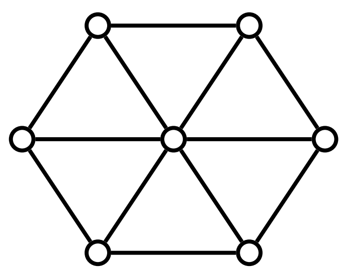
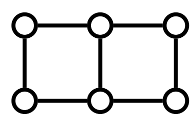
1. Here are a few ways we could try to find spanning trees of a given graph $G = (V, E)$ (that isn't already a tree):

Algorithm 1. Find a loop (or *cycle*) in G, and delete any edge from it. Repeat until G is a tree.

Algorithm 2. Choose any edge in G, and delete it if the resulting graph is still connected; if deleting that vertex would disconnect the graph, choose another edge. Repeat until G is a tree.

Algorithm 3. Pick a *root* vertex $v \in V$. Starting at v, walk across edges in the graph in any way, never returning to a previously visited vertex. If you get stuck, follow your path backwards until you can travel to a previously unvisited vertex. Take all the edges you walked across to be the edges of the tree.

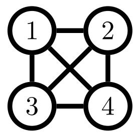
Can you prove that these methods actually give us a spanning tree of G at the end? If not, can you come up with an example where it fails? Do you have a better algorithm (or an improvement on one of the three above)? Which algorithm is the easiest to execute? To get started, try running each algorithm on the following graphs:

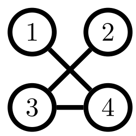
2. Given a graph G, the *degree* of a vertex is the number of edges attached to it. If we package all the degrees of the vertices of a tree in to a list in increasing order, $(d_1, d_2, \ldots, d_n)$, with $d_1 < d_2 < \cdots < d_n$, this is called the *degree sequence*. Write down all trees on 4 and 5 vertices, and compute their degree sequences. For example, there is only one tree on 3 vertices $-$ the 'v' shape tree, with vertex set $\{1, 2, 3\}$ and edges $\{(1, 2), (2, 3)\}-$ which has degree sequence $(1, 1, 2)$.

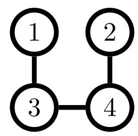
Suppose I tell you that I have drawn a tree T with 6 vertices, and the degree sequence is $(1, 1, 1, 2, 2, 3)$. Can you determine my tree? What if it has 8 vertices, and degree sequence $(1, 1, 1, 1, 1, 1, 3, 5)$? Does there always exist a tree corresponding to a given degree sequence?

3. The *complete* graph K_n is a graph on the vertices $\{1, 2, \ldots, n\}$, where each pair of vertices has an edge between them. Count the number of spanning trees of K_4, shown below:



Many of these spanning trees 'look' the same: that is, we can get from one to the other by switching the labels around. For example, the two spanning trees of K_4 below are the same if we switch the labels 1 and 2:





How many 'different' spanning trees are there in K_4?

(Hard) How many 'different' spanning trees does K_n have? It might help to start by computing this number for small n, and trying to find a pattern.

This is a pretty drawing of a tree.

Did you know: about one third of the United States is covered by trees!